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AN ARBITRARILY DISTORTABLE BANACH SPACE

BY

THOMAS SCHLUMPRECHT

Department of Mathematics, The University of Texas at Austin
Austin, TX 78712, USA

ABSTRACT

In this work we construct a "Tsirelson like Banach space" which is arbitrarily distortable.

1. Introduction

We consider the following notions.

Definition: Let X be an infinite dimensional Banach space, and $\|\cdot\|$ its norm. If $|\cdot|$ is an equivalent norm on X and $\lambda > 1$ we say $|\cdot|$ is a **λ-distortion of X** if for each infinite dimensional subspace Y of X we have

$$\sup\left\{\frac{|y_1|}{|y_2|} : y_1, y_2 \in Y \quad \|y_1\| = \|y_2\| = 1\right\} \geq \lambda .$$

X is called **λ-distortable** if there exists a λ-distortion on X. X is called **distortable** if X is λ-distortable for some $\lambda > 1$, and X is called **arbitrarily distortable** if X is λ-distortable for all $\lambda > 1$. ∎

Remark: R. C. James [3] showed that the spaces ℓ_1 and c_0 are not distortable. Until now these are the only known spaces which are not distortable.

From the proof of [7, Theorem 5.2, p.145] it follows that each infinite dimensional uniform convex Banach space which does not contain a copy of ℓ_p, $1 < p < \infty$, has a distortable subspace. In [2] this result was generalized to any infinite dimensional Banach space which does not contain a copy of ℓ_p, $1 \leq p < \infty$, or c_0.

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A famous open problem (the "distortion problem") is the question whether or not ℓ_p, $1 < p < \infty$, is distortable.

In this paper we construct a Banach space X which is arbitrarily distortable. We first want to mention the following questions which are suggested by the existence of such a space.

PROBLEM: *Is every distortable Banach space arbitrarily distortable? Is, for example, Tsirelson's space T (as presented in [6, Example 2.e.1]) arbitrarily distortable?*

2. Construction of X

We first want to introduce some notations.

The vector space of all real valued sequences (x_n) whose elements are eventually zero is denoted by c_{00}; (e_i) denotes the usual unit vector basis of c_{00}, i.e., $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ if $i \neq j$. For $x = \sum_{i=1}^{\infty} \alpha_i e_i \in c_{00}$ the set $\text{supp}(x) = \{i \in \mathbb{N} : \alpha_i \neq 0\}$ is called the **support of** x. If E and F are two finite subsets of $\mathbb{N}$ we write $E < F$ if $\max(E) < \min(F)$, and for $x, y \in c_{00}$ we write $x < y$ if $\text{supp}(x) < \text{supp}(y)$. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^{\infty} x_i e_i \in c_{00}$ we put $E(x) := \sum_{i\in E} x_i e_i$.

For the construction of X we need a function $f : [1,\infty) \to [1,\infty)$ having the properties (f_1) through (f_5) as stated in the following lemma. The verification of $(f_1),(f_2)$, and (f_3) are trivial while the verification of (f_4) and (f_5) are straightforward.

LEMMA 1: *Let $f(x) = \log_2(x+1)$, for $x \geq 1$. Then f has the following properties:*

(f_1) *$f(1) = 1$ and $f(x) < x$ for all $x > 1$,*

(f_2) *f is strictly increasing to ∞,*

(f_3) *$\lim_{x\to\infty}(f(x)/x^q) = 0$ for all $q > 0$,*

(f_4) *the function $g(x) = x/f(x)$, $x \geq 1$, is concave, and*

(f_5) *$f(x) \cdot f(y) \geq f(x \cdot y)$ for $x, y \geq 1$.*

For the sequel we fix a function f having the properties stated in Lemma 1.

On c_{00} we define by induction for each $k \in \mathbb{N}_0$ a norm $|\cdot|_k$. For $x = \sum x_n \cdot e_n \in c_{00}$ let $|x|_0 = \max_{n\in\mathbb{N}} |x_n|$. Assuming that $|x|_k$ is defined for some $k \in \mathbb{N}_0$ we put

$$|x|_{k+1} = \max_{\substack{\ell\in\mathbb{N} \\ E_1 < E_2 < \cdots < E_\ell \\ E_i \subset \mathbb{N}}} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)|_k .$$

Since $f(1) = 1$ it follows that $(|x|_k)$ is increasing for any $x \in c_{00}$ and since $f(\ell) > 1$ for all $\ell \geq 2$ it follows that $|e_i|_k = 1$ for any $i \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Finally, we put for $x \in c_{00}$

$$\|x\| = \max_{k\in\mathbb{N}} |x|_k \ .$$

Then $\|\cdot\|$ is a norm on c_{00} and we let X be the completion of c_{00} with respect to $\|\cdot\|$.

The following proposition states some easy facts about X.

PROPOSITION 2: (a) (e_i) *is a 1-subsymmetric and 1-unconditional basis of* X; *i.e., for any* $x = \sum_{i=1}^{\infty} x_i e_i \in X$, *any strictly increasing sequence* $(n_i) \subset \mathbb{N}$ *and any* $(\varepsilon_i)_{i\in\mathbb{N}} \subset \{-1,1\}$ *it follows that*

$$\Big\|\sum_{i=1}^{\infty} x_i e_i\Big\| = \Big\|\sum_{i=1}^{\infty} \varepsilon_i x_i e_{n_i}\Big\| \ .$$

(b) *For* $x \in X$ *it follows that*

$$\|x\| = \max\Big\{|x|_0 \ , \sup_{\substack{\ell\geq 2\\ E_1<E_2<\cdots<E_\ell}} \frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|E_i(x)\|\Big\}$$

(where $|x|_0 = \sup_{n\in\mathbb{N}}|x_n|$ *for* $x = \sum_{i=1}^{\infty} x_i e_i \in X$*).*

Proof of Proposition 2: Part (a) follows from the fact that (e_i) is a 1-unconditional and 1-subsymmetric basis of the completion of c_{00} with respect to $|\cdot|_k$ for any $k \in \mathbb{N}_0$, which can be verified by induction for every $k \in \mathbb{N}$.

Since c_{00} is dense in X it is enough to show the equation in (b) for an $x \in c_{00}$. If $\|x\| = |x|_0$ it follows for all $\ell \geq 2$ and finite subsets $E_1, E_2, \ldots, E_\ell$ of $\mathbb{N}$ with $E_1 < E_2 < \cdots < E_\ell$

$$\frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|E_i(x)\| = \max_{k\geq 0}\frac{1}{f(\ell)}\sum_{i=1}^{\ell}|E_i(x)|_k \leq \max_{k\geq 1}|x|_k \leq \|x\| \ ,$$

which implies the assertion in this case.

If $\|x\| = |x|_k > |x|_{k-1} \geq |x|_0$, for some $k \geq 1$, there are $\ell, \ell' \in \mathbb{N}$, $\ell \geq 2$, finite subsets of $\mathbb{N}$, $E_1, E_2, \ldots, E_\ell$ and $E_1', E_2', \ldots, E_{\ell'}'$ with $E_1 < E_2 < \cdots < E_\ell$ and

$E'_1 < E'_2 < \cdots < E'_{\ell'}$, and a $k' \in \mathbf{N}$ so that

$$\begin{aligned}
\|x\| &= |x|_k \\
&= \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)|_{k-1} \\
&\leq \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| \\
&\leq \sup_{\substack{2 \leq \tilde{\ell} \\ \tilde{E}_1 < \tilde{E}_2 < \cdots < \tilde{E}_{\tilde{\ell}}}} \frac{1}{f(\tilde{\ell})} \sum_{i=1}^{\tilde{\ell}} \|\tilde{E}_i(x)\| \\
&= \frac{1}{f(\ell')} \sum_{i=1}^{\ell'} \|E'_i(x)\| \\
&= \frac{1}{f(\ell')} \sum_{i=1}^{\ell'} |E'_i(x)|_{k'} \\
&\leq |x|_{k'+1} \leq \|x\| ,
\end{aligned}$$

which implies the assertion. ∎

Remark: (a) The equation in Proposition 2(b) determines the norm $\|\cdot\|$ in the following sense: If $|\!|\!| \cdot |\!|\!|$ is a norm on c_{00} with $|\!|\!| e_i |\!|\!| = 1$ for all $i \in \mathbf{N}$ and with the property that

$$|\!|\!| x |\!|\!| = \max\left\{ |x|_0 \ , \sup_{\substack{\ell \geq 2 \\ E_1 < E_2 < \cdots < E_\ell}} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |\!|\!| E_i(x) |\!|\!| \right\}$$

for all $x \in c_{00}$, then it follows that $\|\cdot\|$ and $|\!|\!| \cdot |\!|\!|$ are equal. Indeed one easily shows by induction for each $m \in \mathbf{N}$ and each $x \in c_{00}$ with $\#\operatorname{supp}(x) = m$ that $\|x\| = |\!|\!| x |\!|\!|$.

(b) The equation in Proposition 2(b) is similar to the equation which defines Tsirelson's space T [6, Example 2.e.1]. Recall that T is generated by a norm $\|\cdot\|_T$ on c_{00} satisfying the equation

$$\|x\|_T = \max\left\{ |x|_0 \ , \sup_{\substack{\ell \in \mathbf{N} \\ \ell \leq E_1 < \cdots < E_\ell}} \tfrac{1}{2} \sum_{i=1}^{\ell} \|E_i(x)\|_T \right\}$$

(where $\ell \leq E_1$ means that $\ell \leq \min E_1$). Note that in the above equation the supremum is taken over all "admissible collections" $E_1 < E_2 < \cdots < E_\ell$ (meaning that $\ell \leq E_1$) while the norm on X is computed by taking all collections $E_1 < E_2 < \cdots < E_\ell$. This forces the unit vectors in T to be not subsymmetric, unlike in X. The admissibility condition, on the other hand, is necessary in order to imply that T does not contain any ℓ_p, $1 \leq p < \infty$, or c_0, which was the purpose of its construction. ∎

We will show that X does not contain any subspace isomorphic to ℓ_p, $1 < p < \infty$, or c_0 and secondly that X is distortable, which by [3] implies that it cannot contain a copy of ℓ_1 either. Thus, in the case of X, the fact that X does not contain a copy of ℓ_1 is caused by the factor $1/f(\ell)$ (replacing the constant factor $\frac{1}{2}$ in T) which decreases to zero for increasing ℓ.

In order to state the main result we define for $\ell \in \mathbb{N}$, $\ell \geq 2$, and $x \in X$

$$\|x\|_\ell := \sup_{E_1 < E_2 < \cdots < E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| \ .$$

For each $\ell \in \mathbb{N}$, $\|\cdot\|_\ell$ is a norm on X and it follows that

$$\frac{1}{f(\ell)} \|x\| \leq \|x\|_\ell \leq \|x\| \ , \quad \text{for} \quad x \in X \ .$$

THEOREM 3: *For each $\ell \in \mathbb{N}$, each $\varepsilon > 0$, and each infinite dimensional subspace Z of X there are $z_1, z_2 \in Z$ with $\|z_1\| = \|z_2\| = 1$ and*

$$\|z_1\|_\ell \geq 1 - \varepsilon \ , \quad \text{and} \quad \|z_2\|_\ell \leq \frac{1+\varepsilon}{f(\ell)} \ .$$

In particular, $\|\cdot\|_\ell$ is an $f(\ell)$-distortion for each $\ell \in \mathbb{N}$.

Remark: Considering for $n \in \mathbb{N}$ the space $T_{1/n}$ (see for example [1]) which is the completion of c_{00} under the norm $\|\cdot\|_{(T,1/n)}$ satisfying the equation

$$\|x\|_{(T,1/n)} = \max\left\{ |x|_0 \ , \sup_{\ell \leq E_1 < E_2 < \cdots < E_\ell} \frac{1}{n} \cdot \sum_{i=1}^{\ell} \|E_i(x)\|_{(T,1/n)} \right\}$$

for all $x \in c_{00}$ and putting for $x \in T_{1/n}$

$$|\!|\!| x |\!|\!|_{(T,1/n)} = \sup_{E_1 < E_2 < \cdots < E_n} \sum_{i=1}^{n} \|E_i(x)\|_{1/n}$$

E. Odell [8] observed that $|\!|\!| \cdot |\!|\!|_{(T,1/n)}$ is a $c \cdot n$ distortion of $T_{1/n}$ (where c is a universal constant). This observation led the author toward his construction. ∎

In order to show Theorem 3 we will state the following three lemmas, and leave their proof for the next section.

LEMMA 4: *For $n \in \mathbb{N}$ it follows that*

$$\Big\| \sum_{i=1}^{n} e_i \Big\| = \frac{n}{f(n)} .$$

For the statement of the next lemma we need the following notion. If Y is a Banach space with basis (y_i) and if $1 \le p \le \infty$ we say that ℓ_p is **finitely block represented** in Y if for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there is a normalized block $(z_i)_{i=1}^n$ of length n of (y_i), which is $(1+\varepsilon)$-equivalent to the unit basis of ℓ_p^n and we call (z_i) a block of (y_i) if $z_i = \sum_{j=k_{i-1}+1}^{k_i} \alpha_j y_j$ for $i = 1, 2, \ldots$ and some $0 = k_0 < k_1 < \cdots$ in $\mathbb{N}_0$ and $(\alpha_j) \subset \mathbb{R}$.

LEMMA 5: *ℓ_1 is finitely block represented in each infinite block of (e_i).*

LEMMA 6: *Let (y_n) be a block basis of (e_i) with the following property: There is a strictly increasing sequence $(k_n) \subset \mathbb{N}$, a sequence $(\varepsilon_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and for each n a normalized block basis $(y(n,i))_{i=1}^{k_n}$ which is $(1+\varepsilon_n)$-equivalent to the $\ell_1^{k_n}$-unit basis so that*

$$y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n,i) .$$

Then it follows for all $\ell \in \mathbb{N}$

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_\ell\to\infty} \Big\| \sum_{i=1}^{\ell} y_{n_i} \Big\| = \frac{\ell}{f(\ell)} .$$

Proof of Theorem 3: Let Z be an infinite dimensional subspace of X and $\varepsilon > 0$. By passing to a further subspace and by a standard perturbation argument we can assume that Z is generated by a block of (e_i)

CHOICE OF z_1: By Lemma 5 and Lemma 6 one finds $(y_i)_{i=1}^{\ell} \subset Y$, with $y_1 < y_2 < \cdots < y_\ell$ so that $\|y_i\| \ge 1-\varepsilon$, $1 \le i \le \ell$, and so that $\|\sum_{i=1}^{\ell} y_i\| \le \ell/f(\ell)$. Thus, choosing

$$z_1 = \sum_{i=1}^{\ell} y_i \Big/ \Big\| \sum_{i=1}^{\ell} y_i \Big\|$$

it follows that

$$\|z_1\|_\ell \geq \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|y_i\| \Big/ \Big\| \sum_{i=1}^{\ell} y_i \Big\| \qquad \begin{bmatrix} \text{choose } E_i = \text{supp}(y_i) \\ \text{for } i = 1, \ldots, \ell \end{bmatrix}$$
$$\geq 1 - \varepsilon ,$$

which shows the desired property of z_1.

CHOICE OF z_2: Let $n \in \mathbb{N}$ so that $4\ell/n \leq \varepsilon$ and choose according to Lemma 5 normalized elements $x_1 < x_2 < \cdots < x_n$ of Z so that $(x_i)_{i=1}^n$ is $(1+\varepsilon/2)$-equivalent to the unit basis of ℓ_1^n and put

$$z_2 = \sum_{i=1}^{n} x_i \Big/ \Big\| \sum_{i=1}^{n} x_i \Big\| .$$

Now let $E_1, \ldots, E_\ell$ be finite subsets of $\mathbb{N}$ so that $E_1 < E_2 < \cdots < E_\ell$ and so that

$$\|z_2\|_\ell = \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(z_2)\| .$$

We can assume that E_i is an interval in $\mathbb{N}$ for each $i \leq \ell$. For each $i \in \mathbb{N}$ there are at most two elements $j_1, j_2 \in \{1, \ldots, n\}$ so that

$$\text{supp}(x_{j_s}) \cap E_i \neq \emptyset \qquad \text{and} \quad \text{supp}(x_{j_s}) \setminus E_i \neq \emptyset, \qquad s = 1, 2.$$

Putting for $i = 1, 2, \ldots, \ell$

$$\widetilde{E}_i := \cup\{\text{supp}(x_j) : j \leq n \ \text{ and } \ \text{supp}(x_j) \subset E_i\}$$

it follows that $\|E_i(z_2)\| \leq \|\widetilde{E}_i(z_2)\| + 2/n$, and, thus, from the fact that

$$(\widetilde{E}_i(z_2) : i = 1, 2, \ldots, \ell)$$

is a block of a sequence which is $(1+\varepsilon/2)$-equivalent to the ℓ_1^n unit basis, it follows that

$$\|z_2\|_\ell \leq \frac{\ell}{2n} + \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|\widetilde{E}_i(z_2)\| \leq \frac{\varepsilon}{2} + \frac{1+\varepsilon/2}{f(\ell)} \Big\| \sum_{i=1}^{\ell} \widetilde{E}_i(z_2) \Big\| \leq \varepsilon + \frac{1}{f(\ell)} ,$$

which verifies the desired property of z_2. ∎

3. Proof of Lemmas 4, 5 and 6

Proof of Lemma 4: By induction we show for each $n \in \mathbb{N}$ that $\|\sum_{i=1}^n e_i\| = n/f(n)$. If $n = 1$ the assertion is clear. Assume that it is true for all $\tilde{n} < n$, where $n \geq 2$. Then there is an $\ell \in \mathbb{N}$, $2 \leq \ell \leq n$, and there are finite subsets of $\mathbb{N}$, $E_1 < E_2 < \cdots < E_\ell$, so that

$$\Big\|\sum_{i=1}^n e_i\Big\| = \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \Big\|E_j\Big(\sum_{i=1}^n e_i\Big)\Big\|$$

$$= \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \frac{n_i}{f(n_i)} \qquad [\text{where } n_i = \# E_i \text{ and } \textstyle\sum n_i = n]$$

$$= \frac{\ell}{f(\ell)} \sum_{j=1}^{\ell} \frac{1}{\ell} \cdot \frac{n_i}{f(n_i)}$$

$$\leq \frac{\ell}{f(\ell)} \frac{\frac{n}{\ell}}{f(\frac{n}{\ell})} \qquad [\text{Property } (f_4) \text{ of Lemma 1}]$$

$$= \frac{n}{f(\ell) \cdot f(\frac{n}{\ell})}$$

$$\leq \frac{n}{f(n)} \qquad [\text{Property } (f_5) \text{ of Lemma 1}]$$

Since it is easy to see that $\|\sum_{i=1}^n e_i\| \geq n/f(n)$, the assertion follows. ∎

Proof of Lemma 5: The statement of Lemma 5 will essentially follow from the Theorem of Krivine ([4] and [5]). It says that for each basic sequence (y_n) there is a $1 \leq p \leq \infty$ so that ℓ_p is finitely block represented in (y_i). Thus, we have to show that ℓ_p, $1 < p \leq \infty$, is not finitely block represented in any block basis of (e_i). This follows from the fact that for any $1 < p \leq \infty$, any $n \in \mathbb{N}$ and any block basis $(x_i)_{i=1}^n$ of (e_i) we have (use $E_i := \operatorname{supp}(x_i)$ for $i = 1, \ldots, n$ in order to estimate $\|\sum_{i=1}^n x_i/n^{1/p}\|$)

$$\Big\|\frac{1}{n^{1/p}} \sum_{i=1}^n x_i\Big\| \geq \frac{1}{n^{1/p}} \frac{n}{f(n)} = \frac{n^{1-1/p}}{f(n)}$$

and from (f_3). ∎

Proof of Lemma 6: Let

$$y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n,i),$$

for $n \in \mathbb{N}$ and $(y(n,i))_{i=1}^{k_n}$ $(1+\varepsilon_n)$-equivalent to the $\ell_1^{k_n}$ unit basis.

For $x, \tilde{x} \in c_{00}$ and $m \in \mathbb{N}$ with $x < e_m < \tilde{x}$ we will show that

$$(*) \qquad \lim_{n\to\infty} \|x + y_n + \tilde{x}^{(n)}\| = \|x + e_m + \tilde{x}\| ,$$

where

$$\tilde{x}^{(n)} = \sum_{i=m+1}^{\infty} \tilde{x}_i \cdot e_{i+s_n} \qquad \left(\tilde{x} = \sum_{i=m+1}^{\infty} \tilde{x}_i e_i\right)$$

and $s_n \in \mathbb{N}$ is chosen big enough so that $y_n < \tilde{x}^{(n)}$.

This would, together with Lemma 4, imply the assertion of Lemma 6. Indeed, for $\ell \in \mathbb{N}$ it follows from $(*)$ that

$$\begin{aligned}
\frac{\ell}{f(\ell)} &= \Big\|\sum_{i=1}^{\ell} e_i\Big\| \qquad \text{[Lemma 4]}\\
&= \lim_{n\to\infty} \Big\|e_1 + \sum_{i=2}^{\ell} e_{i+n}\Big\| \qquad \text{[subsymmetry]}\\
&= \lim_{n_1\to\infty} \lim_{n\to\infty} \Big\|y_{n_1} + \sum_{i=2}^{\ell} e_{i+n}\Big\|\\
&= \lim_{n_1\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \Big\|y_{n_1} + e_n + \sum_{i=3}^{\ell} e_{i+m}\Big\|\\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \lim_{m\to\infty} \Big\|y_{n_1} + y_{n_2} + \sum_{i=3}^{\ell} e_{i+m}\Big\|\\
&\;\;\vdots\\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_\ell\to\infty} \Big\|\sum_{i=1}^{\ell} y_{n_i}\Big\| .
\end{aligned}$$

In order to prove $(*)$ we show first the following

CLAIM: For $x, y \in c_{00}$, and $n \in \mathbb{N}$, with $x < e_n < y$ and $\alpha, \beta \in \mathbb{R}_0^+$ it follows that

$$\|x + \alpha e_n\| + \|\beta e_n + y\| \le \max\big\{\|x + (\alpha+\beta)e_n\| + \|y\|, \|x\| + \|(\alpha+\beta)e_n + y\|\big\} .$$

We show by induction for all $k \in \mathbb{N}_0$, all $x, y \in c_{00}$, and $n \in \mathbb{N}$, with $\#\operatorname{supp}(x) + \#\operatorname{supp}(y) \le k$, and $x < e_n < y$ and all $q_1, q_2, \alpha, \beta \in \mathbb{R}_0^+$ that

$$\begin{aligned} &q_1\|x + \alpha t_n\| + q_2\|\beta e_n + y\| \\ &\qquad \le \max\{q_1\|x + (\alpha+\beta)e_n\| + q_2\|y\|,\ q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|\}\ . \end{aligned}$$

For $k = 0$ the assertion is trivial. Suppose it is true for some $k \ge 0$ and suppose $x, y \in c_{00}$, $x < e_n < y$ and $\#\operatorname{supp}(x) + \#\operatorname{supp}(y) = k + 1$. We distinguish between the following cases.

CASE 1: $\|x + \alpha e_n\| = |x + \alpha e_n|_0$ and $\|\beta e_n + y\| = |\beta e_n + y|_0$.
If $\|x + \alpha e_n\| = |x|_0$, then

$$q_1\|x + \alpha e_n\| + q_2\|\beta e_n + y\| = q_1\|x\| + q_2\|\beta e_n + y\| \le q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|\ .$$

If $\|\beta e_n + y\| = |y|_0$ we proceed similarly and if $\|x + \alpha e_n\| = \alpha$ and $\|\beta e_n + y\| = \beta$, and if w.l.o.g., $q_1 \le q_2$, it follows that

$$q_1\|x + \alpha e_n\| + q_2\|\beta e_n + y\| = q_1\alpha + q_2\beta \le q_2(\alpha+\beta) \le q_1\|x\| + q_2\|e_n(\alpha+\beta) + y\|\ .$$

CASE 2: $\|x + \alpha e_n\| \ne |x + \alpha e_n|_0$.
Then we find $\ell \ge 2$ and $E_1 < E_2 < \cdots < E_\ell$ so that $E_i \cap \operatorname{supp}(x) \ne \emptyset$ for $i = 1, \ldots, \ell - 1$ and

$$\begin{aligned} &q_1\|x + \alpha e_n\| + q_2\|\beta e_n + y\| \\ &\quad = \frac{q_1}{f(\ell)}\left[\sum_{i=1}^{\ell-1}\|E_i(x)\| + \|E_\ell(x + \alpha e_n)\|\right] + q_2\|\beta e_n + y\| \\ &\quad \le \frac{q_1}{f(\ell)}\sum_{i=1}^{\ell-1}\|E_i(x)\| + \begin{cases} \frac{q_1}{f(\ell)}\|E_\ell(x) + (\alpha+\beta)e_n\| + q_2\|y\| \\ \qquad \text{or} \\ \frac{q_1}{f(\ell)}\|E_\ell(x)\| + q_2\|(\alpha+\beta)e_n + y\| \end{cases} \\ &\quad \text{[By the induction hypothesis]} \\ &\quad \le \max\{q_1\|x + (\alpha+\beta)e_n\| + q_2\|y\|,\ q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|\}\ , \end{aligned}$$

which shows the assertion in this case.

Since in the case that $\|\beta e_n + y\| \ne |\beta e_n + y|_0$ we can proceed like in Case 2 the assertion of the claim is verified and we can move to the proof of the Lemma.

In order to show the equation $(*)$ we first observe that for all $k \in \mathbb{N}_0$, $|x+e_m+\tilde{x}|_k \le \|x+y_n+\tilde{x}^{(n)}\|$ (which is trivial for $k=0$ and follows easily by induction for all $k \in \mathbb{N}_0$) and, thus, that $\liminf_{n\to\infty} \|x+y_n+\tilde{x}^{(n)}\| \ge \|x+e_m+\tilde{x}\|$. Since every subsequence of (y_n) still satisfies the assumptions of Lemma 6 it is enough to show that

$$\liminf_{n\to\infty} \|x+y_n+\tilde{x}^{(n)}\| \le \|x+e_m+\tilde{x}\| .$$

This inequality will be shown by induction for each $k \in \mathbb{N}_0$ and all $x < e_m < \tilde{x}$ with $\#\operatorname{supp}(x) + \#\operatorname{supp}(\tilde{x}) \le k$. For $k=0$ the assertion is trivial. We assume the assertion to be true for some $k \ge 0$ and we fix $x, \tilde{x} \in c_{00}$ with $x < e_m < \tilde{x}$ and $\#\operatorname{supp}(x) + \#\operatorname{supp}(\tilde{x}) = k+1$.

We consider the following three cases:

CASE 1: $\|x+y_n+\tilde{x}\| = |x+y_n+\tilde{x}|_0$ for infinitely many $n \in \mathbb{N}$. Since

$$|x+y_n+\tilde{x}^{(n)}|_0 \le |x+e_m+\tilde{x}|_0 , \qquad n \in \mathbb{N} ,$$

the assertion follows.

CASE 2: For a subsequence (y'_n) of (y_n) we have

$$\|x+y'_n+\tilde{x}\| = \frac{1}{f(\ell_n)} \sum_{i=1}^{\ell_n} \|E_i^{(n)}(x+y'_n+\tilde{x})\|$$

where $\ell_n \uparrow \infty$ and $E_1^{(n)} < E_2^{(n)} < \cdots < E_{\ell_n}^{(n)}$ are finite subsets of $\mathbb{N}$. Since $f(\ell_n) \to \infty$ when $n \to \infty$ the contributions of x and $\tilde{x}^{(n)}$ to $\|x+y'_n+\tilde{x}^{(n)}\|$ is negligible in this case and it follows that

$$\liminf_{n\to\infty} \|x+y_n+\tilde{x}^{(n)}\| = 1 \le \|x+e_m+\tilde{x}\| .$$

Assume now that neither Case 1 nor Case 2 occurs. By passing to a subsequence we can assume

CASE 3: There is an $\ell \ge 2$ so that

$$\lim_{n\to\infty} \left(\|x+y_n+\tilde{x}^{(n)}\| - \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i^{(n)}(x+y_n+\tilde{x}^{(n)})\| \right) = 0$$

where $E_1^{(n)} < \cdots < E_\ell^{(n)}$ are finite subsets of $\mathbb{N}$ with the following properties:
(a) $\operatorname{supp}(x+y_n+\tilde{x}^{(n)}) \cap E_i^{(n)} \ne \emptyset$, $i \le \ell$, and $\operatorname{supp}(x+y_n+\tilde{x}^{(n)}) \subset \bigcup_{i=1}^{\ell} E_i^{(n)}$.

(b) The set $\operatorname{supp}(x) \cap E_i^{(n)}$, $i = 1, \ldots, \ell$, does not depend on n (note that $\operatorname{supp}(x) < \infty$).

(c) There are subsets $\widetilde{E}_1 < \widetilde{E}_2 < \cdots < \widetilde{E}_\ell$ of $\operatorname{supp}(\tilde{x})$ and integers r_n so that $\operatorname{supp}(\tilde{x}^{(n)}) \cap E_i^{(n)} = \widetilde{E}_i + r_n$, for $n \in \mathbb{N}$, (we use the convention that $\phi < E$ for any finite $E \subset \mathbb{N}$).

(d) For $i \leq \ell$ and $1 \leq j \leq k_n$ we have either $\operatorname{supp}(y(n,j)) \subset E_i^{(n)}$ or $\operatorname{supp}(y(n,j)) \cap E_i^{(n)} = \emptyset$.

Indeed, letting for $i \leq \ell$

$$\widetilde{E}_i^{(n)} := \begin{cases} E_i^{(n)} \text{ if } E_i^n \cap \operatorname{supp}(y_n) = \emptyset \\ E_i^{(n)} \setminus \Big(\operatorname{supp}(y(n,t)) \cup \operatorname{supp}(y(n,s))\Big) \\ \qquad \text{where } s := \min\{\tilde{s} : \operatorname{supp}(y(n,\tilde{s})) \cap E_i^{(n)} \neq \emptyset\} \\ \qquad \text{and } t := \max\{\tilde{s} : \operatorname{supp}(y(n,\tilde{t})) \cap E_i^{(n)} \neq \emptyset\} \end{cases}$$

the value $\sum_{i=1}^{\ell} \|E_i^{(n)}(x + y_n + \tilde{x}^{(n)})\|$ differs from $\sum_{i=1}^{\ell} \|\widetilde{E}_i^{(n)}(x + y_n + \tilde{x}^{(n)})\|$ at most by $2\ell/k_n$, which shows that (d) can be assumed.

(e) For $i \leq \ell$ the value

$$q_i := \lim_{n \to \infty} \frac{\#\{j \leq k_n, \operatorname{supp}(y(n,j)) \subset E_i^{(n)}\}}{k_n}$$

exists.

Now we distinguish between the following subcases.

CASE 3A: There are $\ell_1, \ell_2 \in \mathbb{N}$, so that $0 \leq \ell_1 \leq \ell_2 - 1 < \ell_2 \leq \ell$ and

$$\|x + y_n + \tilde{x}^{(n)}\| = \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1 - 1} \|E_i^{(n)}(x)\| + \|E_{\ell_1}^{(n)}(x + y_n)\| + \sum_{i=\ell_1+1}^{\ell_2} \|E_i^{(n)}(y_n)\| \\ + \|E_{\ell_2+1}^{(n)}(y_n + \tilde{x}^{(n)})\| + \sum_{i=\ell_2+2}^{\ell} \|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg]$$

[we put $E_0 = \emptyset$ and $E_{\ell+1} = \emptyset$]. In this case it follows that

$$\|x + y_n + \tilde{x}^{(n)}\| \leq \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1} \|E_i^{(n)}(x)\| + \sum_{i=\ell_1}^{\ell_2+1} \|E_i^{(n)}(y_n)\| + \sum_{i=\ell_2+1}^{\ell} \|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg] \\ \leq \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1} \|E_i^{(n)}(x)\| + 1 + \varepsilon_n + \sum_{i=\ell_2+1}^{\ell} \|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg]$$

[By (d) and the fact that $(y(j,n))_{j=1}^{k_n}$ is $(1+\varepsilon_n)$-equivalent to the $\ell_1^{k_n}$-unit basis]

$$\le \|x + e_m + \tilde{x}\| + \varepsilon_n,$$

which implies the assertion in this case.

Case 3b: There is an $1 \le \ell_1 \le \ell$ so that

$$\|x+y_n + \tilde{x}^{(n)}\| = \frac{1}{f(\ell)}\left[\sum_{i=1}^{\ell_1-1} \|E_i^{(n)}(x)\| + \|E_{\ell_1}^{(n)}(x+y_n+\tilde{x}^{(n)})\| + \sum_{i=\ell_1+1}^{\ell} \|E_i^{(n)}(\tilde{x}^{(n)})\|\right].$$

Then the assertion can be deduced from the induction hypothesis (note, that by (a) and the fact that $\ell \ge 2$ we have that

$$\#\,\mathrm{supp}E_{\ell_1}^{(n)}(x+\tilde{x}^{(n)}) < \#\,\mathrm{supp}(x+\tilde{x}^{(n)})).$$

Case 3c: There is an $\ell_1 < \ell$ so that

$$\|x+y_n+\tilde{x}^{(n)}\| = \frac{1}{f(\ell)}\left[\sum_{i=1}^{\ell_1-1} \|E_i^{(n)}(x)\| + \|E_{\ell_1}^{(n)}(x+y_n)\| + \|E_{\ell_1+1}^{(n)}(y_n+\tilde{x}^{(n)})\| + \sum_{i=\ell_1+2}^{\ell} \|E_i^{(n)}(\tilde{x}^{(n)})\|\right].$$

We can assume that $\mathrm{supp}(x) \ne 0$ and $\mathrm{supp}(\tilde{x}) \ne \emptyset$ (otherwise we are in case 3b). If q_{ℓ_1} (as defined in (e)) vanishes it follows that $\lim_{n\to\infty} \|E_{\ell_1}^{(n)}(x+y_n)\| = \|E_{\ell_1}^{(n)}(x)\|$. Otherwise there is a sequence $(j_n) \subset \mathbb{N}$ with $\lim_{n\to\infty} j_n = \infty$ so that

$$E_{\ell_1}^{(n)}(y_n) = \frac{1}{k_n}\sum_{j=1}^{j_n} y(n,j)$$

and so that

$$\lim_{n\to\infty} \frac{j_n}{k_n} = q_{\ell_1} > 0 .$$

Since the sequence $(E_{\ell_1}^{(n)}(y_n)/q_{\ell_1})_{n\in\mathbb{N}}$ is asymptotically equal to the sequence $(\tilde{y}_n)$ with

$$\tilde{y}_n := \frac{1}{j_n}\sum_{j=1}^{j_n} y(n,j)$$

(note that $(\tilde{y}_n)$ satisfies the assumption of the Lemma) we deduce from the induction hypothesis for some infinite $N \subset \mathbb{N}$ that

$$\begin{aligned}\lim_{\substack{n\to\infty\\ n\in N}} \|E^{(n)}_{\ell_1}(x+y_n)\| &= q_{\ell_1} \lim_{n\to\infty} \left\| E^{(n)}_{\ell_1}\left(\frac{x}{q_{\ell_1}}\right) + \tilde{y}_n \right\| \\ &\le q_{\ell_1} \left\| E^{(n)}_{\ell_1}\left(\frac{x}{q_{\ell_1}}\right) + e_m \right\| \\ &= \|E^{(n)}_{\ell_1}(x) + q_{\ell_1} e_m\|\end{aligned}$$

(recall that $E^{(n)}(x)$ does not depend on n by (b)). Similarly we show for some infinite $M \subset N$, that

$$\lim_{\substack{n\to\infty\\ n\in M}} \|E^{(n)}_{\ell_1+1}(y_n + \tilde{x}^{(n)})\| \le \|q_{\ell_1+1} e_m + \widetilde{E}_{\ell_1+1}(\tilde{x})\| .$$

From the claim at the beginning of the proof we deduce now that

$$\begin{aligned}&\liminf_{n\to\infty} \|x + y_n + \tilde{x}^{(n)}\| \\ &\le \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1-1} \|E^{(n)}_i(x)\| + \|E^{(n)}_{\ell_1}(x) + q_{\ell_1} e_m\| + \|q_{\ell_1+1} e_m + \widetilde{E}_{\ell_1+1}(\tilde{x})\| \\ &\qquad + \sum_{i=\ell_1+2}^{\ell} \|\widetilde{E}_i(\tilde{x})\|\Bigg] \\ &\le \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1-1} \|E^{(n)}_i(x)\| + \sum_{i=\ell_1+2}^{\ell} \|\widetilde{E}_i(\tilde{x})\| \\ &\quad + \max\Big\{\|E^{(n)}_{\ell_1}(x) + e_m\| + \|\widetilde{E}_{\ell_1+1}(\tilde{x})\| \,,\ \|E^{(n)}_{\ell_1}(x)\| + \|e_m + \widetilde{E}_{\ell_1+1}(\tilde{x})\|\Big\}\Bigg] \\ &\qquad [q_{\ell_1} + q_{\ell_1+1} = 1] \\ &\le \|x + e_m + \tilde{x}\| ,\end{aligned}$$

which shows the assertion in this case and finishes the proof of the Lemma.

Note added in proof: Recently the author was able to show that the above constructed space is complementably minimal. This means that every infinite dimensional subspace of X contains an infinite dimensional subspace which is isomorphic to X and complemented in X.

Recently T. Gowers and B. Maurey found independently for every $C > 0$ an equivalent norm on above Banach space X, so that under this norm X does not contain a C-unconditional basic sequence. They, moreover, succeeded in defining a refinement of the construction which does not contain any unconditional basic sequence. ∎

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